

# **Empirical Logic of Finite Automata: Microstatements versus Macrostatements**

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We compare two approaches to the empirical logic of automata. The first, called partition logic (logic of microstatements), refers to experiments on individual automata. The second, the logic of simulation (logic of macrostatements), deals with ensembles of automata.

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## **INTRODUCTION**

The principal goal of this paper is to bring two approaches to the empirical logic of automata into a coherent perspective. Two groups of researchers (K. Svozil and M. Schaller in Vienna, and A. A. Grib and R. R. Zapatrin in St. Petersburg) have approached this issue from different directions. In this joint paper we investigate the connection between these two approaches.

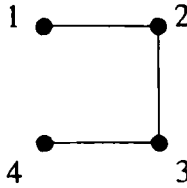
The paper is organized in the following way: in Section 1 the class of automata we are going to deal with is specified. In Section 2 the concept of the empirical logic of automata is specified. The approach taken by the Vienna group based on a partitioning of the set of automaton states is introduced in Section 3, and that of the St. Petersburg group based on a closure operation on the set of inputs is described in Section 4. A comparison of the two approaches concludes the paper.

## **1. AUTOMATON MODEL**

In this paper we deal with the specific class of automata called "normalized" and introduced in Grib and Zapatrin (1990). The main feature of these

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**Fig. 1.** The graph  $P(V, E)$ . The set of vertices  $V = \{1, 2, 3, 4\}$ , the set of edges  $E = \{(1, 2), (2, 3), (3, 4)\}$ .

automata is that they may be completely defined by a nonoriented graph  $P = (V, E)$ ,  $V$  being the set of vertices and  $E$  the set of edges. The states of the automaton correspond to the vertices of the graph  $P$ . We also assume the existence of a so-called final state 0. The inputs of the automaton are in one-to-one correspondence with its states. One could think of applying an input  $v$  as checking the appropriate state. Now assume that the automaton is initially in state  $u$  ( $\neq 0$ ) and that an input  $v$  is applied. Then the *transition*  $\delta: u \rightarrow v$  takes place if and only if the vertices  $u, v$  are linked by an edge of the graph  $P$ , otherwise the automaton goes into the final state 0; i.e.,

$$\delta(u, v) := \begin{cases} v & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$

The two-valued *output function* depends on the *resulting* state. Its value is 0 if the resulting state is the final state, and 1 if the resulting state is any other; i.e.,

$$\lambda(\delta(u, v)) := \begin{cases} 1 & \text{if } \delta(u, v) \neq 0 \\ 0 & \text{if } \delta(u, v) = 0 \end{cases}$$

As an example, consider the normalized automaton associated with the graph  $P$  drawn in Fig. 1. Suppose the automaton is prepared in the initial state corresponding to the vertex 1. Then, if we apply the input 1, the state of the automaton will not change, and it will output the symbol 1. If, starting from the same initial state 1, the input 2 is applied, the state of the automaton changes to 2, through the output value will still be 1. If, on the other hand, we apply any one of the inputs 3 or 4, then the automaton makes a transition to the final state 0 (since there is no edge between the initial vertex 1 and the vertices 3 and 4, respectively) and the output value will be 0. We shall return to this example in the discussion of the propositional structures below.

## 2. EMPIRICAL LOGIC OF AUTOMATA

The motivation behind the investigations in this paper is the construction of primitive *empirical statements* or *propositions* about automata (Finkelstein

and Finkelstein, 1983). Such experimental statements form the basis of the formal investigation of the corresponding logics. In particular, there exist automata for which validation of one empirical statement makes impossible the validation of another empirical statement and vice versa, as first pointed out by Moore (1956).

Thereby, one decisive feature of the setup is the *intrinsic* character of the measurement process: The automaton is treated as a black box with known description but unknown initial state. Automata experiments are conducted by applying an input sequence and observing the output sequence.

In the above example, it is not necessary to input sequences containing more than one symbol, since the nonfinal states are not distinguished by the output function.

### 3. MICROSTATEMENTS: PARTITION LOGIC

The conventional state identification problem (Moore, 1956; Conway, 1971; Braver, 1984) is to obtain information about an unknown initial state. Thereby it is assumed that only *a single* automaton copy is available for inspection. That is, no second, identical example of the automaton can be used for further examination. Alternatively, one may think of it as choosing at random a single automaton from an ensemble of automata which differ only by their initial state. The task then is to find which was the initial state of the chosen automaton.

The partition logic approach to finite automata study was introduced in Svozil (1993) and subsequently treated in Schaller and Svozil (1994, 1995, 1996). There, statements about single automata of the following form  $p_A$  are considered:

$$p_A := \text{“the state } \nu \text{ of the automaton is in } A\text{”} \quad (1)$$

where  $A$  is a subset of the set of automaton states. For any statement  $p_A$  we can build its opposite  $\overline{p_A}$ , namely,

$$p_A^* := \text{“the state } \nu \text{ of the automaton is not in } A\text{”}$$

Let us try to characterize this state identification problem algebraically. Again,  $V$  denotes the set of automaton states. Now associate with any input  $\nu$  the set of automaton states such that any pair of states from any element of the partition are mutually indistinguishable. For normalized automata studied in this paper, this partition reduces to splitting the entire set of states  $V$  into two subsets,

$$V = V_0 \cup V_1 \quad (2)$$

$$V_0 = \{w \in V \mid (\nu, w) \notin E\} \quad (3)$$

$$V_1 = \{w \in V \mid (v, w) \in E\} \tag{4}$$

where  $E$  stands for the set of edges of the graph  $P(V, E)$ . Then, a property  $p_A$  of the form (1) is said to be *testable* if and only if the set  $A$  is contained in the partition (2) associated with some input.

Now consider the entire collection of the sets of the form  $V_0$  and  $V_1$  in (2) for all possible inputs  $v$ . Note that for different inputs, the partitions (2) may coincide (cf. the example below).  $V_0$  may be thought of as the negation of  $V_1$  and vice versa. Therefore, each such pair forms a Boolean algebra  $2^2$ . The propositional structure is then built from these algebras by pasting, i.e., by identifying their greatest and their least elements. The resulting propositional structures are the lattices  $MO_n$ , where  $n$  stands for the number of different partitions (2). The partial order can be interpreted as logical implication " $\Rightarrow$ " on the statements about a *single* automaton from the ensemble. This is why we call these statements *microstatements*.

To take up the example mentioned earlier, consider the partitions produced by all possible inputs  $v = 1, 2, 3, 4$ . They are

$$v = 1: V = \{1, 2\} \cup \{3, 4\} \tag{5}$$

$$v = 2: V = \{1, 2, 3\} \cup \{4\} \tag{6}$$

$$v = 3: V = \{1\} \cup \{2, 3, 4\} \tag{7}$$

$$v = 4: V = \{1, 2\} \cup \{3, 4\} \tag{8}$$

Note that the inputs 1 and 4 produce the same partitions. The resulting propositional structure is drawn in Fig. 2. It should be emphasized that although the elements of the lattice associated with the partition logic are the subsets of the set of states  $V$ , the partial order relation ("implication  $\Rightarrow$ ") is weaker than the set-theoretic inclusion " $\subset$ ." For instance, we can see from the above example that  $\{1\} \subset \{1, 2, 3\}$ , but  $\{1\} \not\Rightarrow \{1, 2, 3\}$ . In general, the partial order relation always implies set inclusion but not vice versa. Only if two statements can be identified by the same experiment, i.e., only if both

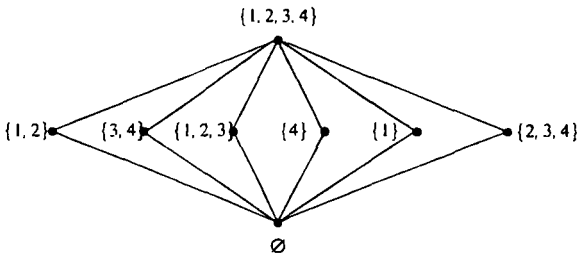


Fig. 2. Lattice  $MO_3$  of the intrinsic propositional calculus.

statements are simultaneously measurable, does the partial ordering coincide with set inclusion.

#### 4. MACROSTATEMENTS: LOGIC OF SIMULATION

Another approach to the logic of finite automata was introduced in Grib and Zapatrin (1990, 1992). Unlike that giving rise to the partition logic described in the previous section, it assumes dealing with ensembles of automata rather than with a single pattern and therefore the notion of property is referred to the *ensemble*. This was the reason to call the statements arising in this approach *macrostatements*.

Before describing the approach in general, we return to our example of an automaton. Suppose now that we have at our disposal two ensembles of such automata. The first is prepared in such a way that some of the constituent automata of the ensemble are in the state  $\{1, 2, 3\}$ . That is, initially, the automata from the first ensemble are either in state 1 or in state 2 or in state 3. The second ensemble is prepared in the state  $\{2, 3, 4\}$ . That is, initially, the automata from the first ensemble are either in state 2 or in state 3 or in state 4.

As before, only one experiment on each particular automaton from the ensemble can be carried out; from each individual automaton in the ensemble, only a single copy is available.

We first perform experiments on the first ensemble. Suppose we have generated the protocol of a sufficiently large sequence of measurements of the results of each input. The values of the output function which can be observed are 0 and 1 for each input.

Next, we deal with the second ensemble in a similar way. Again, looking into the protocol, we see that there are 0's and 1's as the result of each input. Since we have made no particular choice of the distribution of the initial states, we have to conclude that the two states of the ensemble described above are indistinguishable.

However, assume that a third ensemble is prepared in the "pure" state associated with the activation of only the vertex 1. Then, analyzing the protocol, one finds that the output values corresponding to the input 3 are always 0. This makes the protocol differ from the protocols generated from the first two ensembles. Thus, we can distinguish the state of the third ensemble from the states of the first and second ensembles.

We now give the rigorous definitions. With each subset  $A \subseteq V$ , we consider the statement about the ensemble of automata of the following form  $s_A$ :

$$s_A := \text{"the state } v \text{ of each particular automaton is in } A\text{"} \quad (9)$$

For any statement  $s_A$  we can build its opposite  $\overline{s_A}$ , namely,

$$s_A := \text{“the state } \nu \text{ of each particular automaton is not in } A\text{”}$$

Then, a property  $s_B$  of the form (9) is said to be *testable* if and only if there exists a subset  $A \subseteq V$  such that

$$s_B = \overline{s_A}$$

The motivation for this definition is the following. The outcome of the statistical experiment on the ensemble of automata is the protocol of the following form:

input 1	0001110101 ...
input 2	1001010111 ...
...	...
:	:
an input	the values of the output function

The rows of the protocol containing the output values equal to 1 bring no essential information. Only the rows in which the values of the output function are always 0 contain information relevant to our purposes. That is the reason why the subsets having only 0's in the appropriate line of the protocol are treated as testable within this approach.

In our example, the only proper (i.e.,  $\neq 0, V$ ) subsets which are in principle testable, are  $\{1\}$ ,  $\{4\}$ ,  $\{1, 2\}$ ,  $\{3, 4\}$ . For instance, if, on any input of 1, the output is always 0, then one can conclude that the automata in the ensemble are either in (micro-) state 3 or in (micro-) state 4; therefore the ensemble is in the macrostate denoted by  $\{3, 4\}$ . Since the partial order relation strictly follows set inclusion, the lattice of properties looks like the one drawn in Fig. 3.

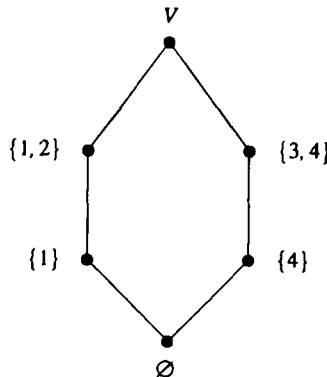


Fig. 3. The diagram of testable properties.

## 5. CONCLUDING REMARKS

Let us state the common features of our two approaches.

*Operationalistic.* Just as in quantum measurements, an extrinsic view is assumed to be inaccessible. All we are allowed to do is to choose the input to be applied and then observe the output values. For instance, in order to measure the electron spin, we are not allowed to “screw open” the electron. Moreover, one cannot in general copy an (arbitrary number of) identical electrons from the single electrons taken from an ensemble.

*Mathematical.* In both approaches, the collection of propositions is associated with a partially ordered set, whose elements are subsets of the set of states of the automaton.

Since our two approaches are based on different empirical setups, there are certain features in which of our two approaches differ.

*Automaton model.* The normalized automata used here are appropriate for the logical simulation of macrostatements. They are assumed to have only two output values. With a different automaton model, to which the partition logic of microstatements is applicable, one obtains more general structures of microstatements than the lattices *MO<sub>n</sub>*. Examples of such automaton models are Moore and Mealy-type automata having more than two output values and thus allowing for more than two elements in the partitions.

### *Operationalistic*

- To formulate the microstatements (to which partition logic is applicable) the experiment is assumed to be carried out with a *single* automaton.
- Macrostatements refer to ensembles of automata.

### *Mathematical*

- In partition logic, the structure of the collection of properties is always a pasting of Boolean algebras. It may not even be a lattice (Svozil, 1993, pp. 137–141).
- The structure of the collection of macroproperties is always a lattice with orthocomplementation. It is not necessarily orthomodular (Zapatin, 1995).
- The collections of subsets associated with testable microstatements and macrostatements do not in general include one another, but always overlap. The partial ordering of these subsets can be induced from both sorts of propositional structures. For macrostatements, the partial order is exactly the set inclusion. As already pointed out, for micro-

statements the set inclusion implies the partial order only if both statements are simultaneously measurable (cf. Section 3).

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